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Coupled oscillators with parity-time symmetry

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ABSTRACT

Different models of coupled oscillators with parity-time (PT) symmetry are studied. Hamiltonian functions for two and three linear oscillators coupled via coordinates and accelerations are derived. Regions of stable dynamics for two coupled oscillators are obtained. It is found that in some cases, an increase of the gain-loss parameter can stabilize the system. A family of Hamiltonians for two coupled nonlinear oscillators with PT-symmetry is obtained. An extension to high-dimensional PT-symmetric systems is discussed.

1. Introduction

A set of coupled oscillators is a basic model of interacting systems. Many important ideas, such as the energy exchange, the eigenfrequency splitting and normal modes, are introduced in the study of this model, see e.g. [1,2].

Many types of linear coupled oscillators can described by a Hamiltonian, which is a quadratic function of coordinates and momenta. Energy is conserved in Hamiltonian systems, therefore conservative systems are associated usually with *ideal* systems without dissipation (loss) and amplification (gain). Let us consider a set of two coupled oscillators:

$$\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \kappa x_2 = 0,$$
$$\ddot{x}_2 - 2\gamma\dot{x}_2 + \omega_0^2 x_2 + \kappa x_1 = 0 \qquad (1)$$

where x_1 and x_2 are the coordinates of the oscillators, γ is the parameter of dissipation (for x_1) and amplification (for x_2), ω_0 is the frequency of a single oscillator, and κ is the coupling parameter. The overdot denotes the derivative on t. Model (1) represents an open system with energy flow from the second oscillator (assuming $\gamma > 0$) to the first oscillator. System (1) has been studied in Refs. [3–6]. Surprisingly, it was found in Ref. [5] that system (1) has the corresponding Hamiltonian.

Model (1) is a simple example of systems with parity-time (PT) symmetry [7,8]. Basically, this means that the system is invariant under inversion of both space and time. If one interchanges x_1 and x_2, and change t to $-t$, Eqs. (1) remains the same. Usually, PT-symmetric systems are stable for some range of the system parameters, and they become unstable when a certain parameter exceed the symmetry breaking threshold [7].

A notion of PT-symmetry came from attempts to extend quantum mechanics beyond Hermitian operators [9]. A typical PT-symmetric Hamiltonian has a complex-valued potential $U(\mathbf{r})$. The imaginary part of $U(\mathbf{r})$ characterizes amplification and dissipation of the wave function. Therefore, a PT-symmetric quantum system is a model with distributed gain and loss. When gain and loss are well balanced, the system is in a stationary state. Later, this idea was expanded to other fields of physics, such as classical mechanics, electronics, and optics. The idea is very promising in optics, where a number of interesting applications has been realized. These include the double refraction, unidirectional light propagation [10–13], perfect absorbers [14] and lasers [15,16].

In present paper, firstly, we present various physical systems that obey PT-symmetry. Then, we consider an extension of model (1), which is Hamiltonian as well. Moreover, we find nonlinear generalization of the model, which is also Hamiltonian, similar to that analyzed in Ref. [17]. Systems of three and more degrees of freedom are also discussed.

2. PT-symmetric models

In this Section, we present several models with PT-symmetry. The aim of this Section is to demonstrate that a PT-symmetric model is not an abstract notion, but it can describe real physical systems. A PT-symmetric system requires an element that provide amplification, or negative dissipation. Such elements are discussed, for example, in Ref. [1], and we use them to construct different types of PT-symmetric systems.

We start with two coupled oscillators shown in Fig. 1(a). Two masses, m_1 and m_2, are connected with each others and walls by

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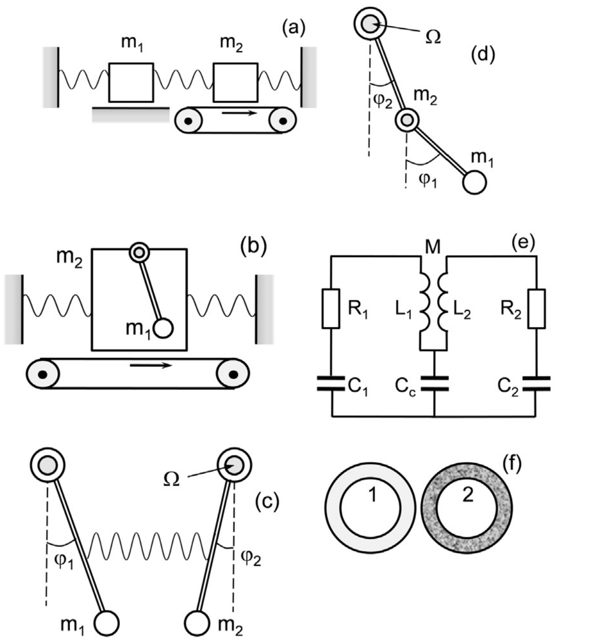

Fig. 1. Examples of PT-symmetric models: (a–d) mechanical systems, (e) an electrical system, and (f) coupled optical waveguides, where the right (darker) waveguide has resonance atoms.

means of springs. Mass m_1 is placed on a fixed frictionless surface, however there is dissipation of energy due to surrounding media. We assume that the dissipation force is proportional to the oscillator velocity. The second mass is placed on a conveyor, which moves with constant velocity V_c. The conveyor drags the mass because of friction. In the absence of coupling between the masses, the equation of motion of m_2 is the following [1]:

$$\ddot{x}_2 + \Gamma \dot{x}_2 + \omega_0^2 x_2 = F(\dot{x}_2 - V_c), \tag{2}$$

where F is the force that depends on the relative velocity of the body and the conveyor. For small velocities $\dot{x}_2$, one can expand $F \approx F(V_c) + F'(-V_c)\dot{x}_2$. A constant force $F(V_c)$ results in a shift of the stationary point for x_2, while the second term results in modification of the dissipation parameter. By a proper choice of $F'(-V_c)$, one can make the dissipation parameter negative, $\Gamma - F'(-V_c) = -2\gamma$, which results in amplification. Then, with a corresponding choice of the system parameters, the model in Fig. 1(a) is reduced to Eqs. (1).

Oscillators in Fig. 1(a) are coupled via coordinates x_1 and x_2. However, it is possible to make inertial coupling between oscillators as shown in Fig. 1(b). Mass m_1 oscillates with dissipation inside m_2. Amplification for mass m_2 is achieved by means of a conveyor, as in Fig. 1(a). One can show that oscillators in this case are coupled via acceleration. (Actually, the equations of motion can be transformed further such that only coordinate coupling remains, however, the initial equations, derived from the Lagrangian, have coupling via acceleration, see e.g. Ref. [2].)

A PT-symmetric mechanical system can be realized with a set of two pendulums coupled via a spring, see Fig. 1(c). Each pendulum has a sleeve on the upper end of the rod. This sleeve is put on a shaft (gray circles in Fig. 1(c)). The shaft of the second pendulum rotates with constant angular frequency Ω. The rotating shaft, similar to the moving conveyor in Figs. 1(a) and (b), introduces amplification for the second pendulum. We mention that a single pendulum with rotating shaft is called the Froude pendulum, see e.g. [1]. In linear approximation, the dynamics of pendulums is described by Eqs. (1). Oscillators in Fig. 1(c) are coupled via coordinates ϕ_1 and ϕ_2, similar to those in Fig. 1(a).

It is possible also to introduce acceleration coupling between the pendulums, as in a double pendulum presented in Fig. 1(d). The first pendulum in Fig. 1(d) is attached to the second one via a movable joint. The second pendulum is the Froude pendulum, so that the rotating shaft provides amplification.

The next example is a pair of oscillatory circuits, presented Fig. 1(e). Two RLC circuits are coupled with each other via mutual inductance M (acceleration coupling) and capacitor C_c (coordinate coupling). The main difference of this circuit from conventional ones is that R_2 has negative resistance, providing gain in the system. Negative resistance can be realized using a tunnel diode or an operational amplifier. PT-symmetric electronic circuits have been studied in Refs. [3,4]. Also, a system of two Josephson junctions with capacitive coupling is modeled by two pendulums coupled via acceleration [18]. Then, it is possible to realize a PT-symmetric system in such superconducting circuits by implementing the negative resistance in either junction.

The last example in this Section is a system of two circular optical waveguides in Fig. 1(f). The waveguides of a size of few microns and less are coupled due to interaction of evanescent fields, therefore it is a coordinate coupling. The second waveguide has resonance atoms inside that can be pumped by external light. This creates amplification in the waveguide, so that with a proper choice of parameters, the system can be considered as PT-symmetric. Similar systems are studied, for example, in Refs. [5,15,16]. The oscillation of electromagnetic fields inside the waveguides is described in linear approximation by equations similar to Eqs. (1).

It is interesting also to consider the quantum behavior of coupled PT-symmetric oscillators. Several examples are presented in Refs. [5,19], see also reviews [7,8]. In Ref. [5], the twofold bifurcation is observed in the classical and quantized versions of the system. In Ref. [19], a relation between a symmetric quadratic Hamiltonian (c.f. Sec. 3) and a pseudo-Hermitian matrix is obtained. We expect that coupling via acceleration can add new features to the dynamics of quantum oscillators.

The examples presented in this Section shows that two types of coupling exist, namely via coordinates and via accelerations. In the next Section, we obtain a model that include both types of coupling.

3. Hamiltonian systems of PT-symmetric oscillators

3.1. Two linear oscillators

In this Section we derive an extension of system (1), which is also Hamiltonian. We start with a general expression for the Hamiltonian written as a quadratic form:

$$H = \mathbf{z}^T \mathbf{A} \mathbf{z}, \tag{3}$$

where $\mathbf{A} = \{a_{ij},\ i, j = 1, \ldots, 4\}$ is a 4×4-matrix, $\mathbf{z} = (x_1, x_2, p_1, p_2)^T$ is a column vector, p_1 and p_2 are canonical momenta of coordinates x_1 and x_2, respectively, and superscript T denote transposition. Without loss of generality, one can assume that $\mathbf{A}$ is symmetric, since a quadratic form with an anti-symmetric matrix equals zero.

The equations of motion are obtained from the Hamilton equations:

$$\dot{x}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, 2. \tag{4}$$

From equations for $\dot{x}_k$, we find the relation between momenta and velocities. Then, equations for $\dot{p}_k$ give the following equations of motion

$$\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \kappa_2 x_2 + \mu_2 \ddot{x}_2 = 0,$$
$$\ddot{x}_2 - 2\gamma\dot{x}_2 + \omega_0^2 x_2 + \kappa_1 x_1 + \mu_1 \ddot{x}_1 = 0, \tag{5}$$

where

$$\gamma = (a_{24} - a_{13}) + (a_{14}a_{33} - a_{23}a_{44})/a_{34},$$
$$\mu_1 = -a_{44}/a_{34}, \quad \mu_2 = -a_{33}/a_{34},$$
$$\omega_0^2 = 4[a_{12}(a_{34}^2 - a_{33}a_{44}) + a_{13}(a_{23}a_{44} - a_{24}a_{34}) + a_{14}(a_{24}a_{33} - a_{23}a_{34})]/a_{34},$$
$$\kappa_1 = 4[a_{13}(a_{13}a_{44} - a_{14}a_{34}) + a_{33}(a_{14}^2 - a_{11}a_{44}) + a_{34}(a_{11}a_{34} - a_{13}a_{14})]/a_{34},$$
$$\kappa_2 = 4[a_{23}(a_{23}a_{44} - a_{24}a_{34}) + a_{33}(a_{24}^2 - a_{22}a_{44}) + a_{34}(a_{22}a_{34} - a_{23}a_{24})]/a_{34}. \tag{6}$$

Equations (5) are an extension of Eqs. (1), they are Hamiltonian, and they involve coupling via both coordinates and accelerations. The only conditions we use are $a_{34} \neq 0$ and $a_{34}^2 - a_{33}a_{44} \neq 0$, therefore model (5) is a most general form of two coupled linear oscillators with PT-symmetry. In particular, each equation in (5) may include only a single gain-loss term.

System (5) is PT-symmetric, because it is invariant with respect to interchange of variables x_1 and x_2, followed by inversion of time. In other words, system (5) is invariant under $\mathcal{PT}$ transformation, where $\mathcal{P}x_1 = x_2$, $\mathcal{P}x_2 = x_1$, $\mathcal{T}x_1(t) = x_1(-t)$, and $\mathcal{T}x_2(t) = x_2(-t)$.

The Hamiltonian of Eqs. (5) is be written as

$$H_L = -\frac{\mu_2 p_1^2 + \mu_1 p_2^2}{2(1-\mu_1\mu_2)} + \frac{p_1 p_2}{1-\mu_1\mu_2} - \gamma(x_1 p_1 - x_2 p_2) + \frac{\kappa_1 + \gamma^2\mu_1}{2} x_1^2 + \frac{\kappa_2 + \gamma^2\mu_2}{2} x_2^2 + (\omega_0^2 - \gamma^2) x_1 x_2. \tag{7}$$

When $\kappa_1 = \kappa_2$ and $\mu_1 = \mu_2 = 0$, we obtain Hamiltonian of Eqs. (1), found in Ref. [5]. The first of the Hamilton equations (4) gives

$$p_1 = \gamma(\mu_1 x_1 - x_2) + \mu_1 \dot{x_1} + \dot{x_2},$$
$$p_2 = \gamma(x_1 - \mu_2 x_2) + \dot{x_1} + \mu_2 \dot{x_2}. \tag{8}$$

Then substitution Eqs. (8) into the second of Eqs. (4) results in Eqs. (5).

Equations (5) depend on six parameters (ω_0, γ, κ_1, κ_2, μ_1, and μ_2). By construction, the oscillators have the same frequency ω_0 and gain-loss parameter γ, however the coupling parameters can be taken unequal. Transforming x_1 or x_2, one can obtain that $\kappa_1 = \kappa_2$, then five independent parameters remain. To simplify the analysis, we assume in the rest of Sec. 3.1 and in Sec. 3.2 that $\kappa_1 = \kappa_2 = \kappa$ and $\mu_1 = \mu_2 = \mu$.

We emphasize an unusual relation between Hamiltonian (7) and motion equations (5). Usually, the Hamiltonian equations (4) for k-th variable (a base variable) give the equation of motion (the base terms) for the same variable. In this equation, the terms that include i-th variable ($i \neq k$) are considered as a coupling. In Eqs. (5), the base terms are considered as a coupling, while the coupling is treated as a base. In other words, the equation for x_1 are obtained from the Hamiltonian equations for x_2, and vise versa. We will see the consequences of this fact in Secs. 3.2 and 3.3.

Typically, PT-symmetric systems are stable when a certain parameter (usually, the gain-loss parameter) is below the threshold [7]. Above the threshold, the symmetry breaking transition occurs that results in instability.

In order to check the stability of motion, we obtain the characteristic equation, by substituting $x_k = a_k \exp(i\omega t)$ into Eqs. (5):

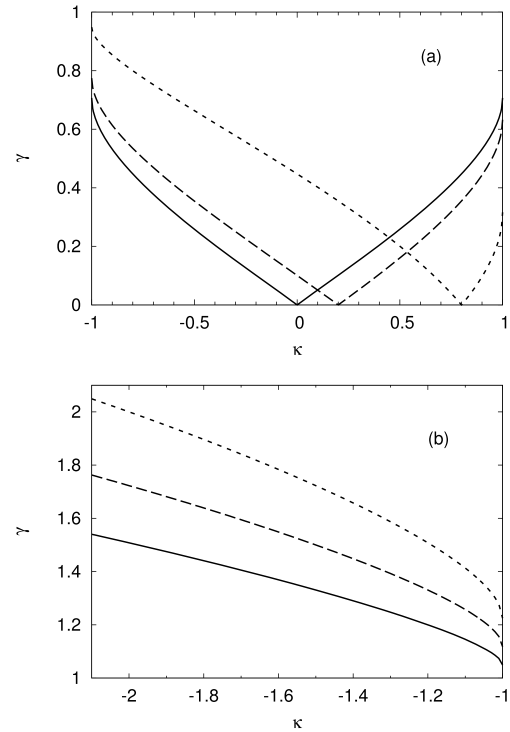

Fig. 2. The diagram of stability: the threshold of γ versus coordinate coupling κ for $\omega_0 = 1$. (a) $\mu = 0$ (solid line), $\mu = 0.2$ (long-dashed line), and $\mu = 0.8$ (short-dashed line). (b) $\mu = 1.2$ (solid line), $\mu = 1.5$ (long-dashed line), and $\mu = 2$ (short-dashed line).

$$(1-\mu^2)\omega^4 + 2(-\omega_0^2 + 2\gamma^2 + \kappa\mu)\omega^2 + \omega_0^4 - \kappa^2 = 0. \tag{9}$$

Solution of this equation gives eigenfrequencies

$$\omega_{1,2}^2 = \frac{1}{(1-\mu^2)}\left[(\omega_0^2 - 2\gamma^2 - \kappa\mu) \pm \sqrt{(\omega_0^2 - 2\gamma^2 - \kappa\mu)^2 - (1-\mu^2)(\omega_0^4 - \kappa^2)}\right], \tag{10}$$

while $\omega_3 = -\omega_1$ and $\omega_4 = -\omega_2$. System (5) is in oscillatory (stable) regime, when all $\omega_{1\text{-}4}$ are real. This is realized, when

$$\mu^2 < 1, \quad \tilde{\kappa}^2 \leq 1, \quad \tilde{\gamma}^2 \leq \tilde{\gamma}_1^2, \quad \text{or}$$
$$\mu^2 > 1, \quad \tilde{\kappa}^2 \geq 1, \quad \tilde{\gamma}^2 \geq \tilde{\gamma}_2^2, \quad \tilde{\kappa}\mu < 0, \tag{11}$$

where

$$\tilde{\gamma}_1^2 = \left[1 - \tilde{\kappa}\mu - \sqrt{(1-\mu^2)(1-\tilde{\kappa}^2)}\right]/2,$$
$$\tilde{\gamma}_2^2 = \left[1 - \tilde{\kappa}\mu + \sqrt{(1-\mu^2)(1-\tilde{\kappa}^2)}\right]/2,$$

where $\tilde{\gamma} = \gamma/\omega_0$ and $\tilde{\kappa} = \kappa/\omega_0^2$.

Fig. 2 shows the diagram of stability for different values of μ. In absence of gain and loss $\tilde{\gamma} = 0$, stable oscillations exist only when $|\mu| < 1$ and $|\tilde{\kappa}| < 1$. When $\tilde{\gamma} > 0$, the stability region for $0 < \mu < 1$, Fig. 2(a), is below the corresponding curve. This is a typical situation: an increase of $\tilde{\gamma}$ results in PT-symmetry breaking and infinite growth of amplitudes. For $\mu > 1$, Fig. 2(b), the unstable dynamics becomes stable, when $\tilde{\gamma}$ is above the corresponding curve. This is an interesting case, in which an increase of the gain-loss parameter plays a stabilizing role in contrast to typical examples. We attribute this type of behavior to the presence of coupling via acceleration.

For $0 < \mu < 1$, there are two branches of the threshold curve, and the threshold equals zero, when $\tilde{\kappa} = \mu$, see Fig. 2(a). For

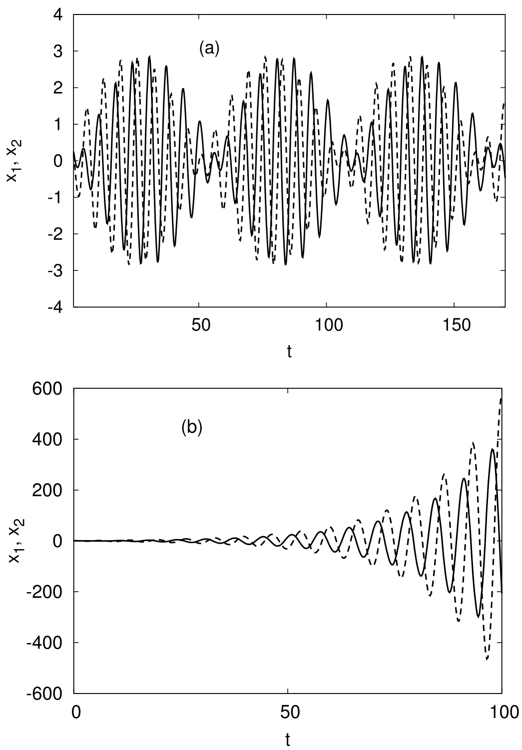

Fig. 3. Numerical solution of Eqs. (5) for $\omega_0 = 1$, $\kappa = 0.5$, $\mu = 0.2$. (a) $\gamma = 0.15$, and (b) $\gamma = 0.17$. Solid (dashed) lines are for $x_1(t)$ $[x_2(t)]$.

$\mu = 1$, only one (left) threshold branch remains. The stability diagram for $\tilde{\gamma} < 0$ (for $\mu < 0$) is obtained by reflection of Fig. 2 with respect to κ-axis (γ-axis).

Fig. 3 shows results of numerical simulations of Eqs. (5) for $\omega_0 = 1$, $\kappa = 0.5$, $\mu = 0.2$ and different values of γ. The threshold for PT-symmetry breaking is $\gamma_{\rm th} = 0.1604$, see Fig. 2(a). There exist stable oscillations in the system below the threshold, see Fig. 3(a). Since the system is characterized by the two normal frequencies, Eq. (10), there is a beating between them. When γ exceeds the threshold, see Fig. 3(b), the system becomes unstable. In Fig. 3(b), one pair of normal frequencies is real, while the other is complex (pure imaginary). With further increase of γ, all ω_k become complex, so that x_1 and x_2 diverge exponentially, without oscillations.

There is a seeming contradiction between the energy conservation and an infinite growth of coordinates and velocities in systems (1) and (5) above the threshold. We mention that an infinite growth of coordinates is not prohibited in Hamiltonian systems. For example, small deviations of the inverted pendulum are described by $\ddot{x} - \omega_0^2 x = 0$ with the Hamiltonian $H = \dot{x}^2/2 - \omega_0^2 x^2/2$. Then, x and $\dot{x}$ diverge, while energy remains constant. Infinite growth of coordinates often means that the model under consideration is incomplete, so that more terms are necessary for an adequate description of the process.

By introducing new variables $u_1 = x_1 + x_2$ and $u_2 = x_2 - x_1$, Eqs. (5) can be written as two oscillators with gain-loss coupling:

$$(1+\mu)\ddot{u}_1 - 2\gamma\dot{u}_2 + (\omega_0^2 + \kappa)u_1 = 0,$$
$$(1-\mu)\ddot{u}_2 - 2\gamma\dot{u}_1 + (\omega_0^2 - \kappa)u_2 = 0. \tag{12}$$

Multiplying (12) by $\dot{u}_1$ and $\dot{u}_2$, respectively, we obtain another form of Hamiltonian (in initial coordinates)

$$H_{L1} = \frac{1}{2}\Big[(1+\mu)(\dot{x}_1 + \dot{x}_2)^2 - (1-\mu)(\dot{x}_2 - \dot{x}_1)^2 + (\omega_0^2 + \kappa)(x_1 + x_2)^2 - (\omega_0^2 - \kappa)(x_2 - x_1)^2\Big] \tag{13}$$

This form of the Hamiltonian does not depend explicitly on γ, while H_L does. Switching to canonical variables (x_k and p_k) transforms H_{L1} to H_L. Yet, Hamiltonian H_{L1} can be useful for construction of Hamiltonian functions for high-dimensional systems.

3.2. Two nonlinear oscillators

We consider here a nonlinear generalization of model (2)

$$\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \kappa x_2 + \mu\ddot{x}_2 + F_1(x_1, x_2) = 0,$$
$$\ddot{x}_2 - 2\gamma\dot{x}_2 + \omega_0^2 x_2 + \kappa x_1 + \mu\ddot{x}_2 + F_2(x_1, x_2) = 0, \tag{14}$$

where F_1 and F_2 are nonlinear functions, such that $F_1(x_1, x_2) = F_2(x_2, x_1)$. In order to make system (14) Hamiltonian, one has to find a potential $U_{NL}(x_1, x_2)$ such that $F_1 = \partial U_{NL}/\partial x_2$ and $F_2 = \partial U_{NL}/\partial x_1$. It easy to find such Hamiltonian by using the same transformation as for Eqs. (12). Clearly, forces F_1 and F_2 should satisfy the following conditions

$$F_1(x_1, x_2) + F_2(x_1, x_2) = 2f_+(x_1 + x_2),$$
$$F_1(x_1, x_2) - F_2(x_1, x_2) = 2f_-(x_2 - x_1), \tag{15}$$

where $f_+(z)$ and $f_-(z)$ are some functions. Then

$$F_1(x_1, x_2) = f_+(x_1 + x_2) + f_-(x_2 - x_1),$$
$$F_2(x_1, x_2) = f_+(x_1 + x_2) - f_-(x_2 - x_1). \tag{16}$$

For such a choice of F_1 and F_2, Hamiltonian is written as

$$H = H_L + U_+(x_1 + x_2) + U_-(x_2 - x_1), \tag{17}$$

where $U_+(z)$ is an arbitrary function and $U_-(z)$ is an even function. For Hamiltonian in Eq. (17), nonlinear forces are found as

$$F_1 = \frac{\partial}{\partial x_2}[U_+(x_1 + x_2) + U_-(x_2 - x_1)],$$
$$F_2 = \frac{\partial}{\partial x_1}[U_+(x_1 + x_2) + U_-(x_2 - x_1)]. \tag{18}$$

We note that F_1 (F_2) is found as a derivative of $U_+ + U_-$ on x_2 (x_1). This is a consequence of unusual relation between the equations of motion and the Hamiltonian.

Let's consider several examples of systems with Hamiltonian (17). One can take U_+ and U_- in the form of polynomial functions

$$U_+ + U_- = a(x_1 + x_2)^m + b(x_2 - x_1)^n, \tag{19}$$

where a, b, m and n are some constants. For $n = m = 4$ and $a = -b = 1/8$, we get Eqs. (14) with $F_1 = (x_1^2 + 3x_2^2)x_1$ and $F_2 = (3x_1^2 + x_2^2)x_2$. Such a model for $\mu = 0$ has been studied in Ref. [17]. We also mention that the reduction of this model, using the rotating wave approximation, results in a system that describes the dynamics of waves in two coupled optical waveguides [17]. The authors of Ref. [17] have proved also the exact solvability of the reduced system.

If we take U_+ and U_- in the form of trigonometric functions

$$U_+ + U_- = a\cos(x_1 + x_2) + b\cos(x_2 - x_1), \tag{20}$$

we obtain a system with nonlinearities similar to coupled pendulums (see Fig. 1(c) and (d)). For example, when $a = -b = -1/2$, we get Eqs. (14) with $F_1 = \sin x_1 \cos x_2$ and $F_2 = \cos x_1 \sin x_2$.

Therefore, in this Section we obtain a family of Hamiltonian functions (17) that correspond to PT-symmetric nonlinear systems with two degrees of freedom.

3.3. Extension to N linear oscillators

There are different ways to generalize Eqs. (5) to a system of N linear oscillators, such that the system is PT-symmetric and Hamiltonian [6]. A simple model is a set of PT-symmetric pairs of oscillators, coupled via coordinates only. A complete model involves coupling of the both types between all oscillators. However, in this case, an explicit equation for Hamiltonian becomes complicated.

Here, we present an example of three coupled PT-symmetric oscillators written as

$$\ddot{x}_1 + 2\gamma\dot{x}_1 + \omega_0^2 x_1 + \kappa_2 x_2 + \kappa_{23} x_3 + \mu_2 \ddot{x}_2 + \mu_{23}\ddot{x}_3 = 0,$$
$$\ddot{x}_2 - 2\gamma\dot{x}_2 + \omega_0^2 x_2 + \kappa_1 x_1 + \kappa_{13} x_3 + \mu_1 \ddot{x}_1 + \mu_{13}\ddot{x}_3 = 0,$$
$$\ddot{x}_3 + \omega_3^2 x_3 + \kappa_{13} x_1 + \kappa_{23} x_2 + \mu_{13}\ddot{x}_1 + \mu_{23}\ddot{x}_2 = 0. \quad (21)$$

The Hamiltonian of Eqs. (21) has the following form

$$H = \frac{1}{2\Delta}[-(\mu_2 - \mu_{23}^2)p_1^2 - (\mu_1 - \mu_{13}^2)p_2^2 + (1 - \mu_1\mu_2)p_3^2] + \frac{1}{\Delta}[(1 - \mu_{13}\mu_{23})p_1p_2 + (\mu_2\mu_{13} - \mu_{23})p_1p_3 + (\mu_1\mu_{23} - \mu_{13})p_2p_3] + \frac{\gamma}{1 - \mu_{13}\mu_{23}}[-x_1p_1 + x_2p_2 + \mu_{13}x_1p_3 - \mu_{23}x_2p_3] \frac{1}{2}\left[\gamma^2\frac{\mu_1 - \mu_{13}^2}{(1 - \mu_{13}\mu_{23})^2} + \kappa_1\right]x_1^2 + \frac{1}{2}\left[\gamma^2\frac{\mu_2 - \mu_{23}^2}{(1 - \mu_{13}\mu_{23})^2} + \kappa_2\right]x_2^2 + \frac{\omega_3^2}{2}x_3^2 + \left(\omega_0^2 - \frac{\gamma^2}{1 - \mu_{13}\mu_{23}}\right)x_1x_2 + \kappa_{13}x_1x_3 + \kappa_{23}x_2x_3, \quad (22)$$

where $\Delta = 1 - \mu_1\mu_2 + \mu_1\mu_{23}^2 + \mu_2\mu_{13}^2 - 2\mu_{13}\mu_{23}$.

Equations (21), similarly to Eqs. (5) and (14), have two types of coupling. Therefore, they generalize the systems considered earlier [5,6]. We note a peculiar structure of the coefficient matrix in Eqs. (21). This is a consequence of the relation, mentioned in Sec. 3.1, between the Hamiltonian and the equation of motions.

The analysis of the coefficients shows that only symmetric configurations are possible. Namely, either all three oscillators are coupled with each other, yielding a symmetric planar configuration, or the oscillators are connected as $1-3-2$, forming one-dimensional structure. An asymmetric configuration, like $1-2-3$, is not possible, because breaking the coupling between oscillators 1 and 3 results in decoupling of oscillators 2 and 3. Also, one cannot form a Hamiltonian PT-symmetric system from three oscillators, where two oscillators are dissipative with γ_1 and γ_2, respectively, and the third oscillator is with gain $-\gamma_3$, even when there is a balance $\gamma_1 + \gamma_2 = \gamma_3$. However, it should be noted that asymmetric gain-loss balanced systems do exist, as discussed in Ref. [20], however they are not Hamiltonian. Therefore, we conjecture that Hamiltonian PT-symmetric systems can have only symmetric configuration.

A general N-dimensional PT-symmetric system, described by the Hamiltonian, can be constructed using the following rules, cf. Eqs. (5) and (21): (i) The i-th equation of motion that corresponds to the i-th Hamiltonian equations may have coordinates x_j and accelerations $\ddot{x}_j$ for all $j = 1, \ldots N$, and all but i-th dissipation/amplification terms $\dot{x}_k$, where $k \neq i$. (ii) Coefficients of cross-coupling terms are symmetric, i.e. $c_{ij} = c_{ji}$, $i \neq j$, while c_{ii} are independent of each other. Here c_{ij} is a common notation for κ_{ij}, γ_{ij} and μ_{ij}. (iii) The basis coordinate in the i-th equation of motion is taken from one of $k \neq i$ (if N is even). When the basis coordinate in the i-th equation is fixed, say k_i, all other terms $\sim \dot{x}_k$, $k \neq k_i$, should be omitted. Rules (ii) and (iii) provide restrictions for possible configurations of PT-symmetric Hamiltonian systems.

4. Conclusion

In this paper, we have described different examples of PT-symmetric systems. These examples provide physical intuition for understanding the properties of PT-symmetric systems. We give more general form of linear Hamiltonian systems with gain and loss that include coupling via both the coordinates and accelerations.

We have found regions of instability for the generalized system of two coupled PT-symmetric oscillators. We have demonstrated that PT-symmetry can play stabilizing role, suppressing instability in a system.

A family of Hamiltonian functions for two nonlinear oscillators has been obtained. The possibility of Hamiltonian PT-symmetric systems in higher dimensions has been discussed. In particular, three coupled linear oscillators are analyzed. We found that it is impossible to make asymmetric Hamiltonian systems with balanced gain and loss.

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