Dissociation of soliton molecules under periodic perturbation in dipolar quantum gases

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Abstract.   In this paper we develop a collective coordinate approach for the dynamics of two solitons, governed by the nonlocal Gross-Pitaevskii equation (GPE). The model adequately describes both the small amplitude and large amplitude dynamics of soliton bound states (also known as soliton molecules), including the dissociation phenomena. Detailed comparison between theoretical predictions and numerical simulations of the governing GPE is presented.

1. Introduction

Solitons as localized nonlinear waves exhibit many interesting properties [1]. In particular, they can form stable bound states known as *soliton molecules* (SM). For the first time SM were experimentally observed in dispersion - managed optical fibers [2]. These soliton compounds may find important applications in fiber optic communication systems to enhance their data carrying capacity [3]. The existence of two-soliton bound states in Bose - Einstein condensates (BEC) with contact atomic interactions and some dynamic phenomena involving SM were reported in [4, 5].

As the solitons are neutral objects, interactions between them in homogeneous optical fibers and ordinary BEC occur indirectly, through modification of the medium properties. Essential point to be stressed here is that, solitons in media with uniform dispersion and local nonlinearity can either attract or repel each other, depending on the phase difference between individual solitons. The interaction between solitons in uniform media occurs only when their waveforms appreciably overlap. In contrast to this, in nonlocal media such as dipolar BEC, solitons can interact even though their waveforms do not overlap.

True molecular type of interaction, when two solitons attract each other at long distances and repel when they are close, was found to exist in dipolar quantum gases. Consequently, at some fixed separation between solitons attraction (due to dipolar forces) and repulsion (due to anti-phase configuration) can balance each other, and this corresponds to stationary state of the molecule. Small displacement of solitons from their stationary positions gives rise to oscillations near the stationary state. In previous works [6, 7] a variational approach (VA) with Gauss-Hermite trial function was employed to find the waveform of a two-soliton molecule and stationary separation between solitons. Despite very good agreement between the VA and numerical simulations of the Gross-Pitaevskii equation (GPE) for small amplitude dynamics, the model [6, 7] becomes inadequate for large amplitude dynamics of the molecule. In particular, it

cannot describe the dissociation phenomena, when the molecule breaks into individual solitons under external forcing or parametric perturbations.

This paper is aimed at developing the collective coordinate approach (CCA) for the dynamics of two-soliton molecules, which can properly describe both the small amplitude and large amplitude oscillations of the molecule, including the dissociation phenomena. In similar setting the CCA for two solitons, governed by the standard nonlinear Schrödinger equation (NLSE), was introduced in Ref. [8]. However, as we mentioned above, standard NLSE does not support stable SM's with finite binding energy. In this work we extend the CCA [8] to soliton bound states in nonlocal GPE and find novel phenomena, not observed in the NLSE. In subsequent sections we present the mathematical model and corresponding numerical simulations.

2. Theory and numerical simulations

The dynamics of a quasi-one dimensional dipolar BEC is governed by the nonlocal GPE, which is written in normalized units as follows

$$
i\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + g(t)\psi \int\limits_{-\infty}^{+\infty} R(|x-z|)\,|\psi(z,t)|^2 dz = 0, \tag{1}
$$

where $\psi(x,t)$ is the mean-field wave function of the condensate, g is the coefficient, responsible for the long - range dipolar atomic interactions. In experiments the strength of dipolar interactions can be made time-dependent $g \to g(t)$ by means of external polarizing magnetic fields [9]. $R(x)$ is the response function, which characterizes the degree of non-locality of the medium. Below we consider the simplest Gaussian form $R(x) = \exp(-x^2/2w^2)/(\sqrt{2\pi}w)$ which is sufficient for qualitative understanding of the properties of dipolar BEC. Also, for simplicity we excluded the contact interactions term $\sim q\,|\psi|^2\,\psi$ from the governing equation. This is justified for condensates of atomic species having large natural magnetic dipole moment, such as ^{52}Cr $(d = 6\mu_B)$, ^{164}Dy $(d = 10\mu_B)$ and ^{168}Er $(d = 7\mu_B)$, with μ_B being the Bohr magneton.

It can be verified, that equation (1) is obtained from the following Lagrangian density

$$
\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 - \frac{g(t)}{2}|\psi(x,t)|^2 \int\limits_{-\infty}^{\infty} R(x-z)|\psi(z,t)|^2 dz. \tag{2}
$$

To develop the collective coordinate approach, we consider two anti-phase Gaussian functions

$$
\psi(x,t) = A\exp\left(-\frac{(x+\xi)^2}{2a^2} + iv(x+\xi) + i\varphi\right) - A\exp\left(-\frac{(x-\xi)^2}{2a^2} - iv(x-\xi) + i\varphi\right). \tag{3}
$$

The time-dependent parameters of individual solitons $A(t)$, $a(t)$, $\xi(t)$, $v(t)$, $\varphi(t)$, meaning the amplitude, width, center-of-mass position, velocity and phase, respectively, are considered as collective coordinates. The averaged Lagrangian $L = \int_{-\infty}^{\infty}\mathcal{L}dx$ is evaluated using Eq. (3)

$$
\begin{aligned}
L = \; & A^2 a\sqrt{\pi}\left[\frac{1}{2a^2} + v^2 + 2v\xi_t + 2\varphi_t - \left(\frac{1}{2a^2} - \frac{\xi^2}{a^4} + 2\xi v_t + 2\varphi_t\right)\exp\left(-\frac{\xi^2 + a^4 v^2}{a^2}\right)\right] - \\
& g(t)\frac{A^4 a^2\sqrt{\pi}}{\sqrt{2(a^2 + w^2)}}\left[1 - 4\cos\left(\frac{a^2 v\xi}{a^2 + w^2}\right)\exp\left(-\frac{a^6 v^2 + 2a^4 v^2 w^2 + 3a^2\xi^2 + 2\xi^2 w^2}{2a^2(a^2 + w^2)}\right) + \right. \\
& \left. \exp\left(-\frac{2\xi^2}{a^2 + w^2}\right) + \exp\left(-\frac{2\xi^2 + 2a^4 v^2}{a^2}\right) + \exp\left(-\frac{2\xi^2(a^2 + w^2) + 2a^4 v^2 w^2}{a^2(a^2 + w^2)}\right)\right]. \tag{4}
\end{aligned}
$$

In deriving the last expression for L we assumed the parameter a to be time-independent, as in Ref. [8]. The norm of the molecule is reduced due to destructive interference of two anti-phase solitons placed at a distance 2ξ from each other: $N_m = \int_{-\infty}^{\infty} |\psi|^2 dx = 2A^2 a \sqrt{\pi}[1 - \exp(-\xi^2/a^2)]$. Obviously, when the solitons are far apart ($\xi \gg a$), the interference is negligible, and the norm of the molecule is twice of the single soliton's norm $N_m = 2N = 2A^2 a \sqrt{\pi}$.

As we have mentioned above, the stationary waveform and small amplitude dynamics of SM is well described by VA [6,7]. For a given set of input parameters w, g_0, N_m, the width a_m of a SM can be evaluated from the algebraic equation [6]

$$a_m^4(a_m^4 + 4w^4)(a_m^2 + w^2)^{-7/2} - 4\sqrt{2\pi}/(g_0 N_m) = 0. \tag{5}$$

This equation describes the bond length of a two-soliton molecule $\Delta = 4a_m/\sqrt{\pi}$ as a function of its norm N_m. The wave function for SM $\psi_m = A_m x \exp(-x^2/2a_m^2)$ with a norm $N_m = A_m^2 a_m^3 \sqrt{\pi}/2$, used in [6], can be represented by two anti-phase Gaussians (3) with parameters

$$A = [2N_m/(a_m \sqrt{\pi} e)]^{1/2}, \quad a = a_m e/4, \quad \xi = a_m, \text{ where } e = 2.71828... \tag{6}$$

In the left panel of figure 1 we compare the shape of a two-soliton molecule $\psi_m(x)$ found from equation(5) and two anti-phase Gaussians with parameters according to equation(6). Numerically "exact" waveform constructed using the Nijhof's method [10], is also presented. Apparently, the results of VA and Nijhof's method almost coincide.
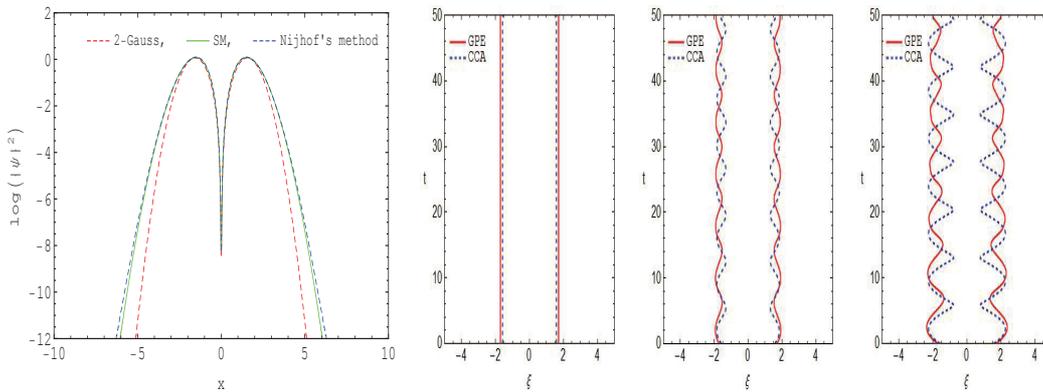
Figure 1. Left panel: The shape of a SM (green solid line) found from equation (5) using $w = 5$, $g_0 = 20$, $N_m = 4$, compared with two anti-phase Gaussians (3) with $a = 1.032$, $A = 1.046$, $\xi = 1.519$, $v = 0$, $\varphi = 0$ (red dashed line). Numerically "exact" waveform is constructed by Nijhof's method (blue dashed line). Right panels: Oscillations of solitons when they are set in motion in opposite directions with initial velocity (from left to right): $v = 0$, $v = 0.2$ and $v = 0.5$, according to GPE (red solid line) and CCA (blue dashed line).

Once the averaged Lagrangian (4) is available and the stationary state of the SM is found, its dynamics under parametric perturbations, such as time dependent strength of dipolar interactions $g(t)$, can be explored using the coupled system of Euler-Lagrange equations for collective coordinates $\eta \rightarrow A(t)$, $\xi(t)$, $v(t)$, $\varphi(t)$

$$\frac{d}{dt}\frac{\partial L}{\partial \eta_t} - \frac{\partial L}{\partial \eta} = 0. \tag{7}$$

In the right panels of figure 1 we show the dynamics of the molecule when the solitons are set in motion in opposite directions with some initial velocity. In the unperturbed molecule ($v = 0$) the solitons stay motionless at their equilibrium positions. If the molecule is excited due to initial velocity ($v > 0$), the solitons start to oscillate. This is the evidence of the molecular type of interaction between solitons in dipolar BEC. Another important aspect to be stressed here is that, collisions between dipolar solitons are strongly inelastic, which shows up as a phase shift after each collision.

In figure 2 we present typical examples of the dynamics of a two-soliton molecule under periodically varying strength of dipolar interactions. As can be seen from this figure, the developed model adequately describes the oscillations of solitons due to parametric perturbation, and quite accurately predicts the time instance, at which the molecule dissociates.
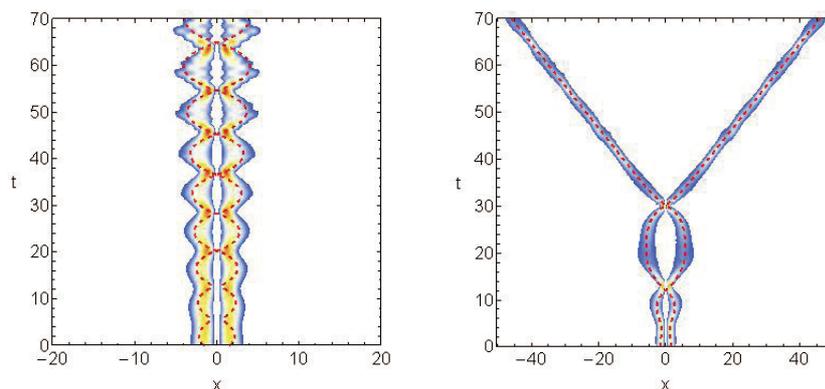
Figure 2. Dynamics of a two-soliton molecule under periodically varying strength of dipolar interactions $g(t) = g_0 (1 + \varepsilon \sin(\omega t))$ according to CCA (red dashed line) and GPE (density plot $|\psi|^2$). Left panel: The solitons vibrate under weak perturbation $\varepsilon = 0.1$ with frequency $\omega = 0.756$. Right panel: At stronger perturbation $\varepsilon = 0.63$ with the same frequency the molecule decays into freely moving solitons at $t \sim 30$. The parameter values are the same as in figure 1.

3. Conclusions

We have developed a collective coordinate approach for the dynamics of two bright solitons, moving and colliding in dipolar quantum gases. The advantage of the present model over previously reported ones is that, it can adequately describe both the small amplitude and large amplitude dynamics of soliton molecules, including the dissociation phenomena, when the molecule decays into separate freely moving solitons. Theoretical predictions are compared with numerical simulations of the governing GPE, and good agreement between them is found.

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