

Scattering of a two-soliton molecule by Gaussian potentials in dipolar Bose–Einstein condensates

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Abstract

Two bright solitons in a dipolar Bose–Einstein condensate (BEC) can form stable bound states, known as soliton molecules. In this paper we study the scattering of a two-soliton molecule by external potential, using the simplest and analytically tractable Gaussian potential barriers and wells, in one spatial dimension. Collisions of soliton molecules with single solitons are investigated, the latter playing the role of a localized defect. Due to the long-range character of dipolar forces solitons interact with each other even though their waveforms do not appreciably overlap. This is an essentially different feature of dipolar solitons compared to their counterparts in BECs with contact atomic interactions. The result of scattering significantly depends on the potential's strength and velocity of collision. For weak potentials and/or low velocity the molecule preserves its coherence, meantime the internal modes are excited. Scattering by strong potentials at moderately high velocity ends up with dissociation of the molecule. The theoretical model is based on the variational approximation for the nonlocal Gross–Pitaevskii equation (GPE). Predictions of the mathematical model are compared with numerical simulations of the nonlocal GPE, and good qualitative agreement between them is demonstrated.

Keywords: soliton, dipolar interactions, scattering, bound state

(Some figures may appear in colour only in the online journal)

1. Introduction

Bose–Einstein condensation (BEC) of atomic species with large natural magnetic moment, such as chromium [1], dysprosium [2] and erbium [3], has been one of the most important advances in the physics of quantum gases for the last decade. Long-range dipole–dipole atomic interactions, scaled at large distances as $\sim 1/x^3$, principally contribute to the physical properties of these quantum gases. The existence of robust nonlinear excitations in dipolar BEC, called matter wave solitons, was reported in a number of publications [4–6]. Solitons in dipolar BEC exhibit markedly different behavior in mutual collisions [7, 8] and scattering by external potentials, compared to solitons in ordinary BEC with contact

interactions. In contrast to ordinary solitons, dipolar solitons interact with each other even though their waveforms do not appreciably overlap.

Recently a great deal of attention has been paid to scattering and splitting of solitons by external potentials, motivated by applications in atom interferometry [9, 10]. While the problem has been studied in the context of solitons in ordinary BEC with contact atomic interactions [11], scattering of dipolar solitons by external potentials remains less explored.

Solitons emerge from a fine balance between dispersive spreading and nonlinear self focusing of the localized wave. During the interactions solitons exhibit both particle-like and wave-like properties. In particular, two colliding solitons

exchange not only their velocities, as occurs with classical particles, but also their physical locations, which is inherent to quantum particles due to the tunneling phenomenon. Experimental demonstration [12] and theoretical validation [13] of the last property has been a notable achievement in the physics of solitons. The discovery of stable bound states of optical solitons, known as soliton molecules [14–16], represents another significant advance in the field. Recently the existence of similar soliton complexes in quasi-one-dimensional dipolar BEC was reported in [17, 18]. The potential of interaction between dark solitons in dipolar BEC, formation of bound states and some dynamic properties of dark soliton molecules, including their mutual collisions, were investigated in [19, 20].

In this work we study the scattering of a two-soliton molecule by potential barriers and wells to reveal new properties inherent to multi-soliton complexes in dipolar BEC. Of particular interest is the excitation of internal modes of soliton molecules as a result of the scattering event. Internal vibrations of solitons and multi-soliton complexes can significantly affect the outcome of their interaction with localized inhomogeneities, or mutual collisions. For instance, the rate of energy emission by a kink of the ϕ^4 model [21] and bound states of two sine-Gordon kinks, so called wobblers [22], strongly depend on the presence of internal vibrations prior to the impact.

We address the problem using analytical methods and numerical simulations. The analytical method is based on the variational approximation [23, 24] for a two-soliton molecule, which is briefly described below. The corresponding problem for a single soliton scattering in dipolar BEC, confined to a parabolic trap, was explored in [25].

2. The model and variational approach

The governing equation of our model is the one-dimensional nonlocal Gross–Pitaevskii equation (GPE), which in dimensionless units has the following form [6, 25]:

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - V(x)\psi + q|\psi|^2\psi + g\psi\int_{-\infty}^{+\infty}R(|x-\xi|)|\psi(\xi,t)|^2d\xi = 0, \quad (1)$$

where $\psi(x,t)$ is the mean-field wave function of the condensate, $V(x)$ is the external potential, q and g are, respectively, the coefficients of nonlinearity, responsible for the short-range contact and long-range dipole–dipole interactions between atoms in the condensate. When both of these coefficients are negative ($q < 0$, $g < 0$), which corresponds to repulsive contact and repulsive dipolar interactions, equation (1) does not support bright solitons and molecules. In the opposite situation ($q > 0$, $g > 0$) bright solitons and molecules exist, but the role of contact interactions is nonessential, and we shall consider only the case of dipolar attraction ($q = 0$, $g > 0$). More interesting is the situation of

competing nonlinearities ($q \cdot g < 0$), which has been analyzed for single bright solitons in [6].

The wave function is normalized to a reduced number of atoms in the condensate $N = \int_{-\infty}^{+\infty}|\psi(x)|^2dx$, which is a conserved quantity of equation (1). The response function $R(x)$ characterizes the degree of nonlocality of the medium. It shows how strongly the properties of the medium at a given location depend on the properties in its neighborhood. For the contact interactions, when the particles influence each other only when they are at the same spatial point, the response function is equal to the Dirac delta function. For long-range dipole–dipole interactions, the response function was derived in a single mode approximation [26]:

$$R_s(x) = (1 + 2x^2)e^{x^2}\operatorname{erfc}(|x|) - \frac{2}{\sqrt{\pi}}|x|, \quad (2)$$

where $\operatorname{erfc}(x)$ is the complimentary error function. However, this expression features a cusp at $x = 0$, and relevant analytic calculations are complicated. Another function was proposed in [6], which behaves smoothly at the origin and is more convenient for analytical treatment:

$$R(x) = \frac{\delta^3}{(x^2 + \delta^2)^{3/2}}. \quad (3)$$

The value of the cutoff parameter $\delta = \pi^{-1/2}$ is found from the condition of equal areas beneath the corresponding curves $\int_{-\infty}^{\infty}R(x)dx = \int_{-\infty}^{\infty}R_s(x)dx$. A very good correspondence between the above two response functions can be observed from their graphics (see figure 1 in [6]).

To develop the variational approximation (VA) we note that the governing GPE (1) can be generated from the following Lagrangian density:

$$\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 + V(x)|\psi|^2 - \frac{1}{2}q|\psi|^4 - \frac{1}{2}g|\psi(x,t)|^2\int_{-\infty}^{\infty}R(x-\xi)|\psi(\xi,t)|^2d\xi. \quad (4)$$

An important step in the development of the VA is the selection of a suitable trial function. As a trial function for the two-soliton molecule in *anti-phase* configuration we use the first Gauss–Hermite function [27–29]

$$\psi(x,t) = A(t)(x-\xi)\exp\left[-\frac{(x-\xi)^2}{2a(t)^2} + ib(t)(x-\xi)^2 + iv(t)(x-\xi) + i\phi(t)\right], \quad (5)$$

where $A(t)$, $a(t)$, $b(t)$, $\xi(t)$, $v(t)$, $\phi(t)$ are time-dependent variational parameters, meaning the amplitude, width, chirp parameter, position of the center-of-mass, velocity and phase of the soliton molecule, respectively. The velocity is defined as a time derivative of the molecule’s center-of-mass position $v = \dot{\xi}$. The norm $N = A^2a^3\sqrt{\pi}/2$, which is a conserved quantity of the governing equation, is proportional to the number of atoms in the condensate. In fact the waveform (5) can be well approximated by two anti-phase Gaussian functions (single solitons). The spatial separation between

two solitons' center-of-mass positions, which is the analog of a molecular bond length, is defined as $\Delta = 4a/\sqrt{\pi}$. Below we consider the scattering of two-soliton molecules by Gaussian potential barriers ($V_0 > 0$) and wells ($V_0 < 0$):

$$V(x) = V_0 \exp\left(-\frac{x^2}{2d^2}\right). \quad (6)$$

Scattering of molecules by potentials, which are too narrow ($d \ll a$) or too wide ($d \gg a$) compared to the bond length of the molecule, can be regarded as interaction with point-like impurities and smooth potential landscapes, respectively. In such situations unusual scattering phenomena are not expected. Below we consider a more interesting situation when the waist of the potential and the molecular bond length are comparable ($d \sim a$). The analytic approach is aimed at deriving the coupled system of equations for the width and center-of-mass position of the molecule.

Now using the response function (3) and ansatz (5), we evaluate the Lagrangian density (4). Subsequent integration over the space variable $L = \int \mathcal{L} dx$ yields the averaged Lagrangian

$$\begin{aligned} \frac{L}{N} = & \frac{3}{2} a^2 b_t - \frac{1}{2} \xi_t^2 + \phi_t + \frac{3}{4a^2} + 3 a^2 b^2 \\ & + 2\sqrt{2} V_0 d G(a, \xi) - \frac{3qN}{8\sqrt{2}\pi a} - \frac{3gN}{8\sqrt{2}\pi a} F(a), \end{aligned} \quad (7)$$

where

$$\begin{aligned} F(a) = & \mathcal{U}\left(\frac{1}{2}, 0, \frac{1}{2\pi a^2}\right) - \frac{1}{3\pi a^2} \mathcal{U}\left(\frac{3}{2}, 1, \frac{1}{2\pi a^2}\right) \\ & + \frac{1}{4\pi^2 a^4} \mathcal{U}\left(\frac{5}{2}, 2, \frac{1}{2\pi a^2}\right), \end{aligned}$$

$$G(a, \xi) = \frac{d^2(a^2 + 2d^2) + a^2 \xi^2}{(a^2 + 2d^2)^{5/2}} \exp\left(-\frac{\xi^2}{a^2 + 2d^2}\right),$$

$$\mathcal{U}(a, b, c) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ct} t^{a-1} (t+1)^{b-a-1} dt$$

is the confluent hypergeometric function [30], with $\Gamma(a)$ being the gamma function.

The Euler–Lagrange equations $d/dt(\partial L/\partial \nu_t) - \partial L/\partial \nu = 0$ for variational parameters $\nu \rightarrow a, b, \xi, \phi$ give the following coupled system for the width and center-of-mass position of the soliton molecule:

$$\begin{aligned} a_{tt} = & \frac{1}{a^3} - \frac{4\sqrt{2} V_0 d}{3} \frac{\partial G(a, \xi)}{\partial a} \\ & - \frac{qN}{4\sqrt{2}\pi a^2} - \frac{gN}{4\sqrt{2}\pi a^2} H(a), \end{aligned} \quad (8)$$

$$\begin{aligned} \xi_{tt} = & -4\sqrt{2} V_0 d \xi \frac{a^4 + a^2(d^2 - \xi^2) - 2d^4}{(a^2 + 2d^2)^{7/2}} \\ & \times \exp\left(-\frac{\xi^2}{a^2 + 2d^2}\right), \end{aligned} \quad (9)$$

where

$$\begin{aligned} H(a) = & \mathcal{U}\left(\frac{1}{2}, 0, \frac{1}{2\pi a^2}\right) - \frac{3}{2\pi a^2} \mathcal{U}\left(\frac{3}{2}, 1, \frac{1}{2\pi a^2}\right) \\ & + \frac{7}{4\pi^2 a^4} \mathcal{U}\left(\frac{5}{2}, 2, \frac{1}{2\pi a^2}\right) - \frac{5}{8\pi^3 a^6} \mathcal{U}\left(\frac{7}{2}, 3, \frac{1}{2\pi a^2}\right). \end{aligned}$$

From these equations it is evident, that when the molecule is far from the location of the potential ($\xi \gg d$), the dynamics of the center-of-mass and width become independent. Namely, the center-of-mass freely moves with a constant velocity ($\xi_{tt} = 0$), and the width remains at equilibrium value $a = a_0$, or oscillates with a constant frequency ω_0 , if perturbed. The frequency of small amplitude oscillations can be found by expanding equation (8) with $V_0 = 0$ around the fixed point $a(t) = a_0 + a_1(t)$, $a_1 \ll a_0$, which leads to the harmonic oscillator equation $\ddot{a}_1 + \omega_0^2 a_1 = 0$. Although the calculations are simple, the expression for the frequency appears to be cumbersome, therefore we do not present it here. Decoupling of equations (8) and (9) occurs due to vanishing of the action of the potential at large distances ($\xi \gg d, a$), since $\exp[-\xi^2/(a^2 + 2d^2)] \rightarrow 0$, and as a consequence, $G(a, \xi) \rightarrow 0$.

To check the validity of the developed model, we introduce the waveform (5) with parameters, corresponding to the stationary state of equation (8) into GPE (1) and propagate in time. For all parameter settings, where the existence of molecules is predicted by VA, we observed steady propagation of two-soliton molecules. Small amplitude oscillations of solitons near equilibrium positions is attributed to the approximate nature of the variational method. In figure 1 we illustrate a typical waveform of a two-soliton molecule in a pure dipolar condensate ($q = 0, g = 2$) as predicted by VA. For comparison, the waveform is provided which is constructed by Nijhof's method [31], adapted for nonlocal GPE. As an initial waveform for Nijhof's method the profile predicted by VA has been employed. A slow change of the strength of contact interactions according to the law $q(t) = -3 \tanh(\gamma t)$ with $\gamma = 0.01$, while keeping the dipolar interactions constant $g = 2$, gives rise to dissociation of the molecule into separate solitons at $t \sim 20$ (corresponding to $q \approx -0.6$), as shown in figure 2. Further enhancement of the repulsive contact interactions leads to the decay of individual solitons, which is consistent with a threshold behavior of solitons in dipolar BEC with competing local and nonlocal atomic interactions, as reported in [6].

Stability of the solution molecule was checked by adding a weak spatial perturbation into the initial waveform, and its subsequent propagation by GPE (1). The perturbed soliton molecule sheds off some amount of radiation in the form of linear waves (which are absorbed at domain boundaries), and acquires a regular shape. Long lived internal vibrations of the molecule remain after the transient period, which is the specific feature of dipolar soliton molecules. This is the evidence of true stability of the soliton molecule, unlike the conjecture from linear stability analysis, which was shown to be not always valid [32].

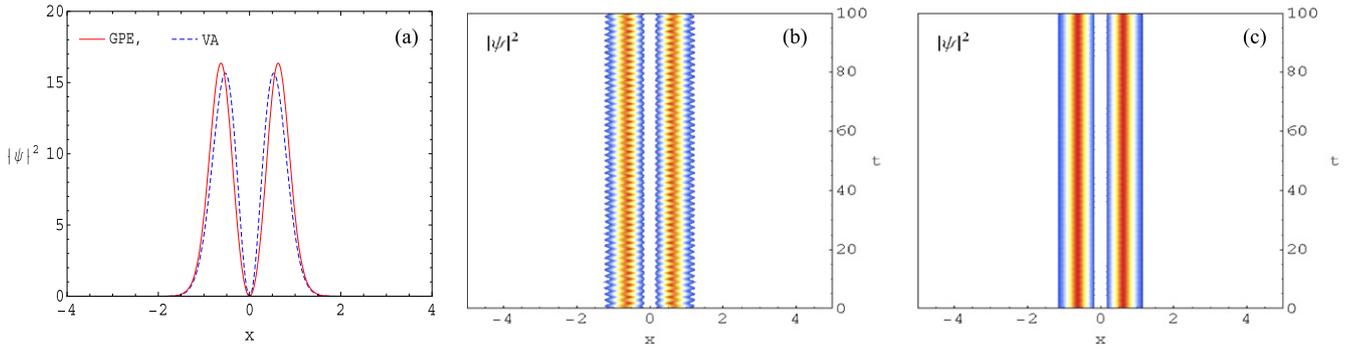


Figure 1. (a) The shape of a two-soliton molecule predicted by VA for $N = 20$, $q = 0$, $g = 2$ (blue dashed line) is compared with the wave form constructed by Nijhof’s method applied to GPE (red solid line). (b) Evolution of the VA predicted solution according to GPE (1) with $V(x) = 0$. Small amplitude oscillations of solitons near equilibrium positions with period $T_0 = 2\pi/\omega_0 = 1.41$ are due to the approximate nature of the VA solution. (c) Stable propagation of the soliton molecule, constructed by Nijhof’s method.

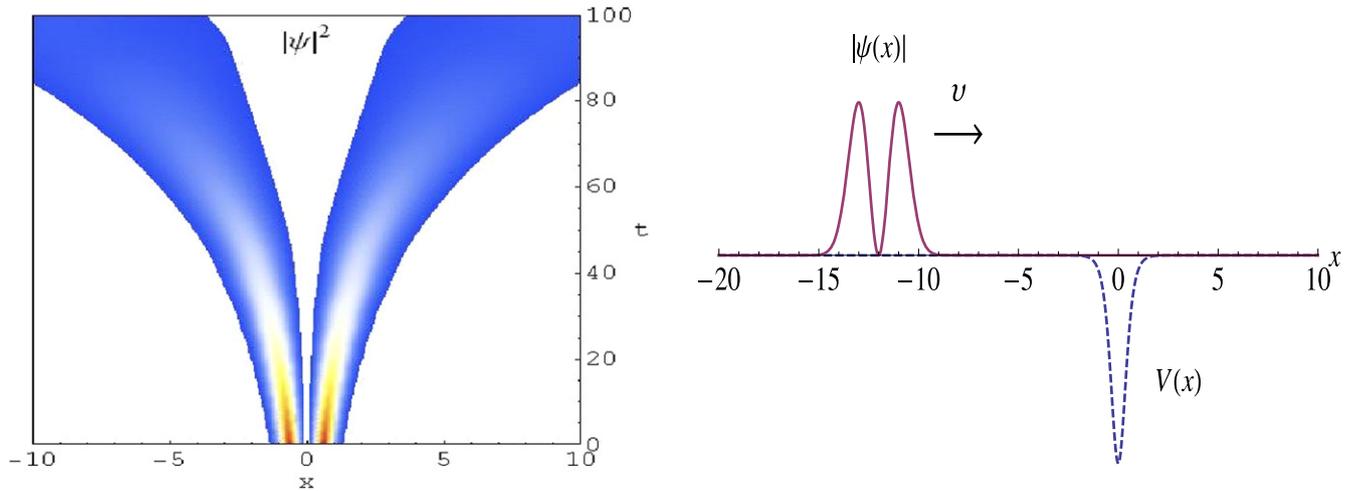


Figure 2. Gradual change of the repulsive contact interactions from zero to $q = -3$, at a fixed value of the attractive dipolar interactions $g = 2$, leads to dissociations of the molecule, and subsequent decay of solitons. The waveform found by Nijhof’s method is employed as the initial state of the molecule.

Figure 3. A sketch of numerical experiments. A two-soliton molecule is set in motion towards the external potential $V(x)$ with some initial velocity v . Depending on the strength of the potential and initial velocity, the molecule can be reflected, trapped or transmitted through the potential. If the potential is sufficiently strong and the velocity is high, the molecule can break up into individual solitons after the scattering event.

Before proceeding with numerical experiments, we comment on the existence of two-soliton molecules in *in-phase* configuration. To form a bound state in this setting, the usual attraction between two in-phase solitons must be balanced by their repulsion due to dipolar forces. However, this configuration appears to be unstable against merging or diverging of solitons. In qualitative terms this can be explained as follows. Each soliton of the molecule moves in the potential created by the other soliton, defined by the nonlocal term of the GPE (1). For the in-phase configuration solitons appear to be positioned on top of a Gaussian-like potential barrier. The instability of interacting soliton pairs located on repulsive potentials was studied in [29, 33].

In the following sections we shall consider the scattering of a soliton molecule by weak and strong potential barriers and wells. A sketch of numerical experiments is illustrated in figure 3. By comparing the predictions of VA with the results of numerical simulations of the GPE (1) one can find the accuracy of the developed mathematical model.

3. Weak potentials

The external potential is considered to be weak if its strength is much less than the chemical potential of the soliton $V_0 \ll \mu$. In ordinary BEC with contact interactions, the chemical potential is given through the soliton amplitude [34] $\mu \sim |\psi(0)|^2$. Therefore, to judge on the potential’s effect we need to define the amplitude of individual solitons A_s , forming the molecule. The maxima of the molecule’s waveform (5) are located at $x = \pm a$, which can be found from the condition $d|\psi|/dx = 0$ at $\xi = 0$. Then the amplitude of individual soliton is

$$A_s = Aae^{-1/2} = \sqrt{\frac{2N}{ae\sqrt{\pi}}}, \quad \text{where } e = 2.71828 \dots \tag{10}$$

For instance, the amplitude of solitons, forming the molecule with norm $N = 20$ and width $a = 0.53$ (see figure 1), is equal

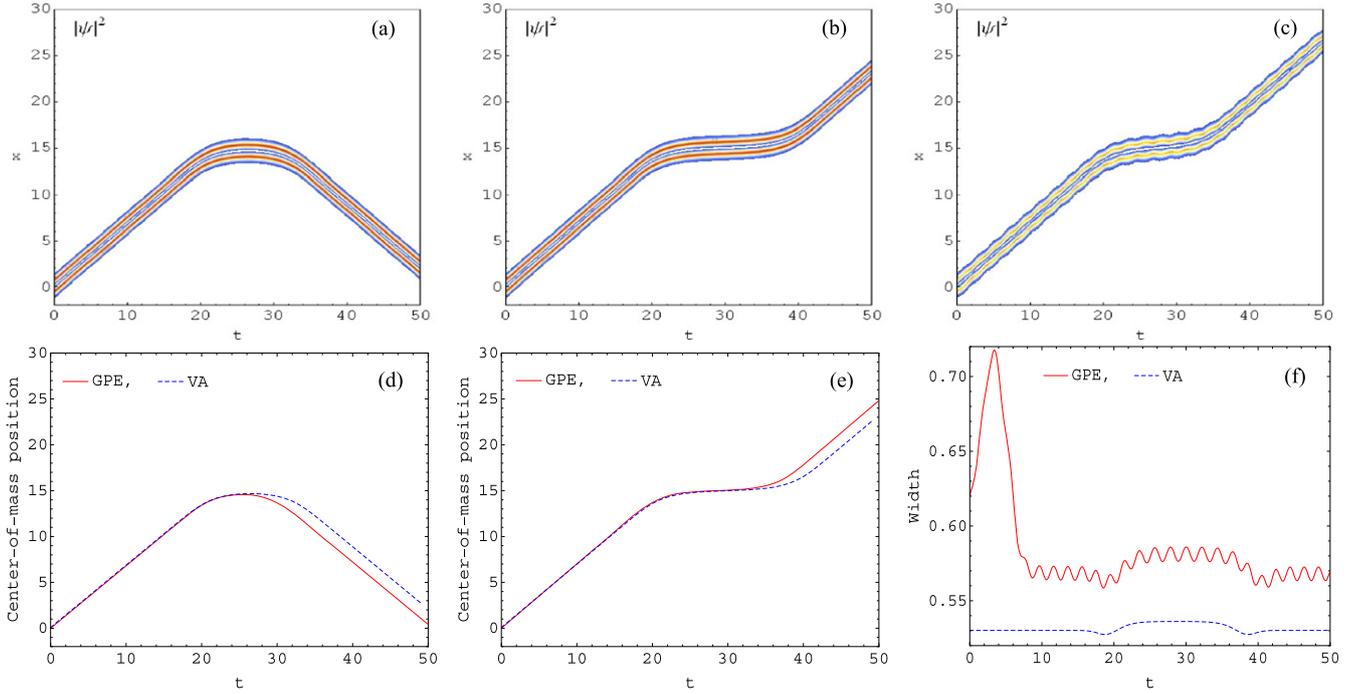


Figure 4. Scattering of a two-soliton molecule by a weak potential barrier (6) with $V_0 = 0.3$ and $d = 1$, located at $x_0 = 15$, for different velocities. Full reflection at velocity $v = 0.68$ (a) and transmission at greater velocity $v = 0.69$ (b). If internal vibrations of the molecule are present prior to collision, the transmission time for the same velocity $v = 0.69$ becomes shorter (c). Dynamics of the center-of-mass position (d), (e) and width (f), according to GPE and VA. Initial parameters are the same as in figure 1.

to $A_s = 3.957$. These values are found from the fixed point of equation (8) at $V_0 = 0$, for pure dipolar condensate ($q = 0, g = 2$).

In figure 4 we illustrate the scattering of a two-soliton molecule by a weak potential barrier located at $x = 15$. At low velocity the molecule exhibits full reflection, while at greater velocity it is transmitted through the potential. After the interaction with the potential barrier a periodic exchange of a small amount of matter between solitons takes place, which is caused by the collision induced phase shift. The effect is more pronounced for stronger potential barriers and greater collision velocities. Deviation of the relative phase between solitons from π is responsible for the exchange of matter between them [18]. This is not accounted for by the trial function (5), and therefore matter exchange cannot be described by the developed VA. Comparison with predictions of the VA shows that scattering by a weak potential slightly perturbs the molecule by exciting its internal modes. The agreement between VA and GPE is sufficiently good for the center-of-mass dynamics, while for the width the agreement is only qualitative. The VA underestimates the width of the molecule by $\sim 10\%$, as can be seen in figure 4(f) after a transient period. These parameters are retrieved from the results of GPE simulation $\psi(x, t)$ as follows:

$$\begin{aligned} \xi(t) &= \frac{1}{N} \int_{-\infty}^{\infty} x |\psi(x, t)|^2 dx, \\ a(t) &= \left(\frac{2}{3N} \int_{-\infty}^{\infty} (x - \xi(t))^2 |\psi(x, t)|^2 dx \right)^{1/2}. \end{aligned} \quad (11)$$

Extensive numerical experiments have shown that scattering on weak potential barriers and wells mainly results in perturbation of the molecule by exciting its internal modes and inducing small matter exchange between solitons. The same pertains to smooth potential barriers and wells, when the variation of the potential on the spatial scale of the molecule is small. In these limits the VA gives reliable predictions for the dynamics of soliton molecules.

4. Strong potentials

In accordance with the definition given in the previous section, by strong potential barriers and wells we presume settings when the strength of the potential is comparable or greater than the amplitude of individual solitons. In addition, the spatial extent of the potential is commensurate with that of the molecule $d \sim a$.

At low velocity of the soliton molecule, incident upon the strong potential barrier, the character of scattering is similar to the case of a weak potential considered in the previous section. Namely, the molecule preserves its coherence after reflection from the potential, while oscillations of solitons near their equilibrium position are excited, and small matter exchange between them is induced. The dynamics of the center of mass and width can still be described by VA on a qualitative level. Below we study scattering of a soliton molecule by strong potential barriers and wells, at moderately large velocities. Since the molecule undergoes significant deformation during the interaction process, and can

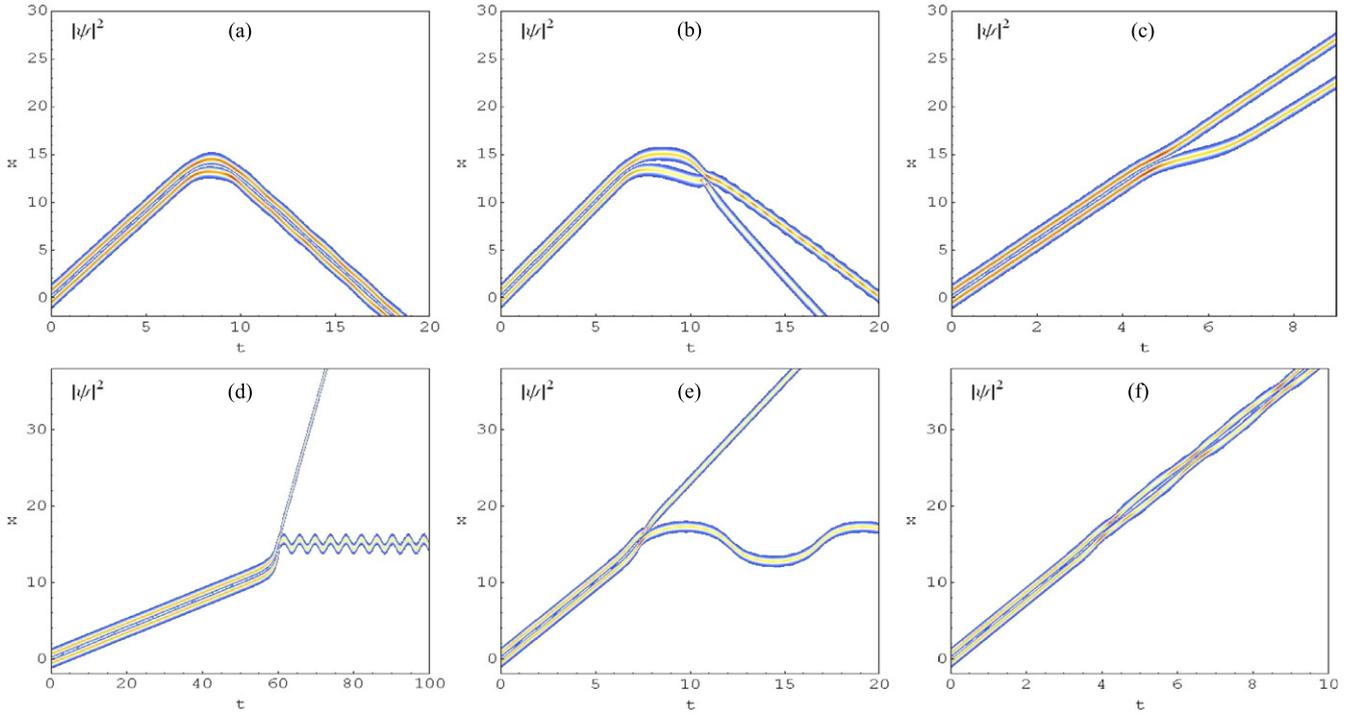


Figure 5. Scattering of a two-soliton molecule by a strong potential barrier $V_0 = 3$ (upper row) and well $V_0 = -3$ (lower row) with spatial extent $d = 1$, at different velocities, according to GPE (1). (a) The molecule is reflected at velocity $v = 1.8$, preserving its coherence. (b) At velocity $v = 2$ the reflected molecule separates into unequal solitons, propagating at different speeds. (c) At greater velocity $v = 3$, the molecule passes through the barrier and disintegrates into equal co-propagating solitons. (d) Low energy collision with a potential well at $v = 0.2$ leads to disintegration of the molecule. One of the solitons is trapped by the well, while the other is transmitted, receiving greater velocity. (e) For collision at velocity $v = 2$ the interaction time is small, so that transmitted soliton has approximately the same velocity as before the collision. (f) For high energy collision with $v = 4$ the interaction time is short and the molecule coherently passes the potential well. The internal mode is excited.

eventually disintegrate, agreement between GPE and VA is not expected in these conditions. Therefore, our studies in this section will be mainly based on numerical experiments.

In figure 5 we illustrate the main characteristic features of scattering of a two-soliton molecule by strong potential barriers and wells. The result of scattering significantly depends on the duration of interaction. To estimate the interaction time we note that the length scales, associated with the soliton molecule and external potential, are given by $l_s = 2a$ and $l_p = d$, respectively. The greater of these two parameters defines the interaction time of the soliton molecule with the potential. In numerical simulations we use the potential whose waist is less ($l_p < l_s$) or comparable with the bond length of the molecule ($l_p \sim l_s$). The result of scattering depends on the ratio between the interaction time of the soliton molecule with a narrow potential $t_{\text{int}} \sim 2a_0/v$ and nonlinear relaxation time $T_0 = 2\pi/\omega_0$. The characteristic velocity follows from the relation $t_{\text{int}} \sim T_0$, and is given by $v_c = a_0\omega_0/\pi$. For parameter settings of the soliton molecule, shown in figure 1 $a_0 = 0.53$, $\omega_0 = 4.45$ the dimensionless characteristic velocity is $v_c = 0.75$. When collision with a potential well occurs at a velocity range less than or comparable with v_c , the molecule breaks up. In such a process, one of the solitons is trapped by the well, while the other leaves the potential area with increased velocity, as shown in figure 5(d). Here the additional kinetic energy is provided through the nonlocal

interactions by the soliton falling into the potential well. If the interaction time is sufficiently short, the molecule undergoes simple disintegration when the leaving soliton has almost the same kinetic energy as prior to collision, as shown in figure 5(e). In both cases of potential wells and barriers, collisions at high velocity ensure short interaction time, and the soliton molecule can retain its coherence after the scattering event (see figure 5(f)).

5. Collision of soliton molecules with single solitons

The collision of soliton molecules with single solitons is relevant to scattering phenomena. In this case the single soliton may play the role of a localized defect, which can move and nonlinearly interact with the incident molecule during the scattering process. There is an essential difference between solitons in ordinary BEC with contact atomic interactions, and those in BEC with long-range dipole-dipole interactions. In the former case solitons interact only when their waveforms overlap, while in the latter case solitons can interact at much greater separation, where the overlap is vanishing. This is due to the long-range character of dipolar forces acting between atoms in the condensate. In turn, the long-range attractive force between solitons leads to their notable acceleration while approaching each other. That is

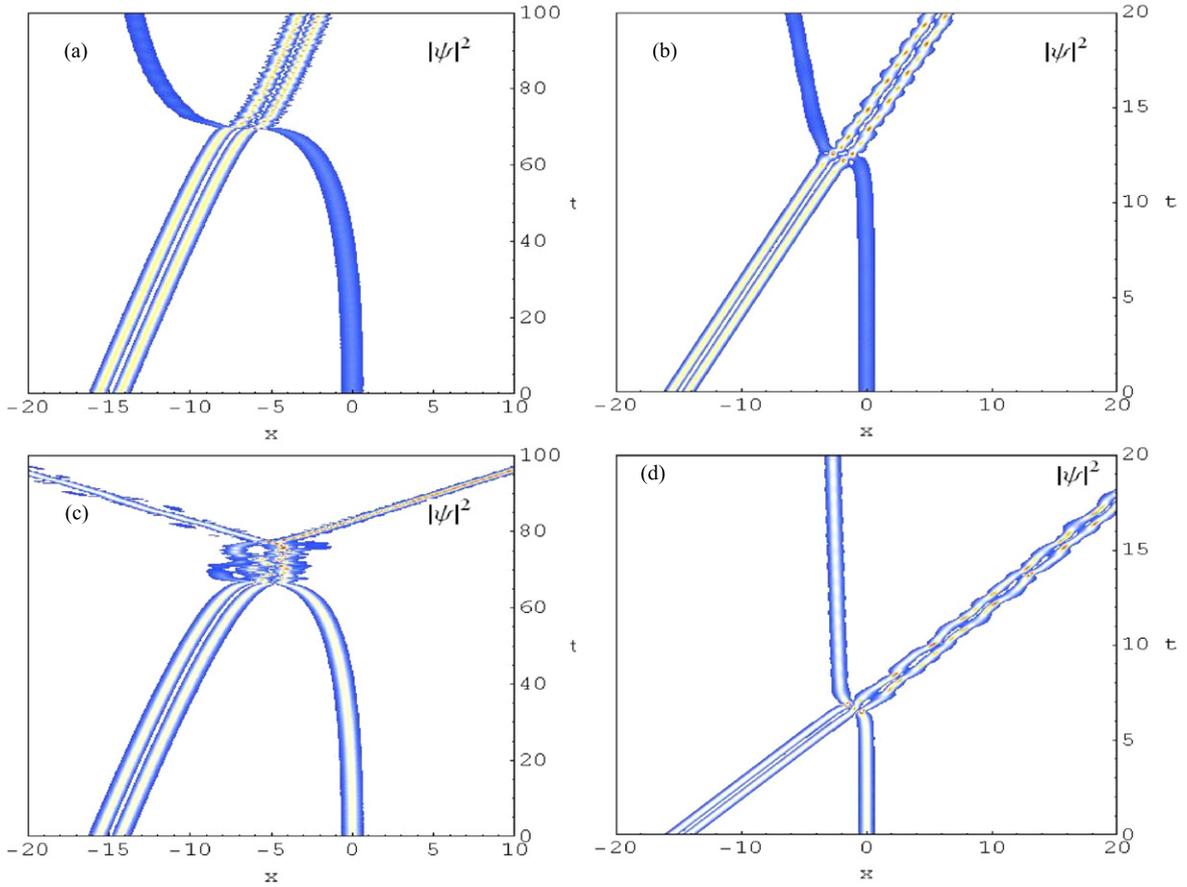


Figure 6. Collision of a moving two-soliton molecule of norm $N = 20$ with a static single soliton located at $x = 0$, in purely dipolar BEC ($q = 0, g = 2$). (a) The single soliton has smaller norm $N_s = 4$, amplitude $A_s = 1.928$, and width $a_s = 0.607$. The velocity of the incident molecule is $v = 0.1$. (b) Collision of the same pair at greater velocity $v = 1$. (c) Collision of the molecule at small velocity $v = 0.1$ with a single soliton of comparable norm $N_s = 10$, $A_s = 3.783$, $a_s = 0.394$. The molecule is destroyed. (d) Collision of the same pair at greater velocity $v = 2$. Both the molecule and single soliton survive after the collision.

why the collision of dipolar solitons in free space occurs with a velocity exceeding some critical value. The minimal collision velocity v_m can be estimated from the binding energy of a two-soliton molecule $v_m \sim \sqrt{|U(a_0)|}$, where the potential $U(a)$ and fixed point a_0 are defined by equation (8).

In figure 6 we show typical collision scenarios between a two-soliton molecule incident on a single soliton located at the origin. A lightweight single soliton starts to move with acceleration towards the approaching molecule and passes through without significant distortion. After the scattering event a high frequency internal vibration of the molecule is excited, as shown in figure 6(a). Faster collision of the same pair ensures a short interaction time. The single soliton is shifted to smaller distance from the origin and acquires relatively small velocity. In both cases the molecule and single soliton survive after the collision. Periodic matter exchange between individual solitons of the molecule is initiated (see figure 6(b)). If the norms of the molecule and single soliton are comparable, the outcome of the collision is different. At slow collision, accordingly long interaction time, the molecule is destroyed. Fragments in the form of single solitons and linear waves leaving the interaction region at high velocity may appear, as shown in figure 6(c). In contrast

to this, after fast collision both the single soliton and the molecule preserve their coherence. The internal modes of the molecule are excited, accompanied by periodic matter exchange between the solitons (see figure 6(d)). From the numerical simulations we conclude that collisions of soliton molecules with single solitons are non-destructive if the single soliton is lightweight (has small norm) or collision time is sufficiently short.

6. Conclusions

The scattering of a two-soliton molecule by Gaussian potential barriers and wells, described by the nonlocal Gross–Pitaevskii equation, has been studied by a variational approximation and numerical simulations. Interaction with weak potentials or lightweight single solitons gives rise to excitation of internal modes of the molecule and leads to matter exchange between its solitons. When the potential is strong and collision velocity is moderately high, the molecule can break up. At sufficiently high collision velocity the molecule preserves its coherence, because during short interaction periods the nonlinear relaxation and rearrangement

processes cannot fully develop. It has been pointed out that unlike solitons in ordinary BEC, dipolar solitons interact with each other even though their waveforms do not appreciably overlap. For attractive dipolar interactions, solitons in free space collide with a velocity exceeding some critical value, which depends on the molecule's binding energy. Predictions of the developed model, based on the variational approach, corroborate the results of numerical simulations of the governing equation. An important subject for future work would be the phenomenon of quantum reflection of dipolar solitons and molecules from potential wells. So far this phenomenon has been explored for solitons and two-soliton bound states in BEC with contact interactions [35–38]. Long-range dipolar interactions may substantially modify the process of quantum reflection.

Acknowledgments

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